EXPONENTIAL MOMENTS OF CANONICAL PHASE: HOMODYNE MEASUREMENTS

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Abstract

A method for direct sampling of the exponential moments of canonical phase from the data recorded in balanced homodyne detection is presented. Analytical expressions for the sampling functions are shown which are valid for arbitrary states. A numerical simulation illustrates the applicability of the method and compares it with the direct measurement of phase by means of double homodyning.

1 Introduction

In the study of the problem of phase of a quantum harmonic oscillator, such as a radiation field, one usually proceeds in one of two different ways. In the first, the phase is defined from the requirement that phase and photon number should be complementary quantities. This first-principle definition leads to the *canonical phase* related to the one sided unitary phase operator [1, 2]. In the second way, phase quantities are defined from the output observed in phase-sensitive measurements, such as eight-port homodyne detection [3]. It was found that in such a scheme the Q function is measured [4]. The *measured phase* distribution can then be obtained by radial integration of the Q function. Whereas in the classical limit the measured phase coincides with the canonical phase, in the quantum regime the two phases significantly differ from each other in general. The most important difference is that the integrated Q function yields a broader and less structurized distribution than the canonical phase [5]. Is there a way which would bring together the advantages of these two approaches - i.e., the theoretical elegance and pronounced structure typical of the canonical phase and the advantage of experimental availability as in the case for the integrated Q function?

Before addressing the problem in more detail, let us mention the concept of *direct sampling* of a quantity from experimental data. This concept has been studied extensively in connection with the density matrix reconstruction [6]. Assume a balanced homodyne measurement of a field quadrature $x(\vartheta)$, where ϑ is the phase of the local oscillator (LO). Performing such measurements with different ϑ on large ensembles of identically prepared states we obtain probability distributions $p(x,\vartheta)$. A quantity $\mathcal{A}$ can be sampled from the homodyne data if it can be expressed as a two fold integral of the measured distribution,

$$\mathcal{A} = \int_{2\pi} d\vartheta \int_{-\infty}^{\infty} dx,\; K_{\mathcal{A}}(x,\vartheta)\, p(x,\vartheta), \tag{1}$$

where $K_{\mathcal{A}}(x,\vartheta)$ is an integration kernel. The quantity $\mathcal{A}$ can represent, e.g., density matrix elements, mean values of operators, etc.

Let us turn to the question of whether the canonical phase distribution can be directly sampled. In this case the quantity $\mathcal{A}$ in Eq. (1) is the phase probability, $\mathcal{A} = p(\varphi)$. In Ref. [7] it was suggested to obtain the exact phase distribution as the limit of a convergent sequence of appropriately parametrized (smeared) distributions each of which can directly be sampled from the homodyne data. The exact phase distribution can then be obtained asymptotically to any degree of accuracy, if the sequence parameter is chosen such that smearing is suitably weak. However, it turned out that whereas the method works well for states with low photon numbers, with higher excitation the smearing parameter must be chosen very small and the corresponding kernels become more and more structurized. This makes sampling problematic for highly excited states. We can see a discrepancy between the quantum and classical regions: on one hand phase can easily be measured in classical physics, on the other hand sampling of the phase distribution becomes tedious for states from the classical region. Can a unified approach be found which would bridge the gap between these two regions and which would enable us to measure the canonical and classical phase distributions on the same footing?

In this contribution we show the possibility of direct sampling of the exponential phase moments of the canonical phase. The corresponding kernels are well behaved functions which, for large $|x|$, approach their classical counterparts. Since the moments contain the same information about the phase properties as the probability distribution itself, the method can serve as a way for experimental determination of the canonical phase.

2 Integration kernels

The canonical phase distribution $p(\varphi)$ of a state $\hat{\varrho}$ is defined as $p(\varphi) = (2\pi)^{-1}\ \langle\varphi|\hat{\varrho}|\varphi\rangle$, where the phase states $|\varphi\rangle$ are [2] $|\varphi\rangle = \sum_{n=0}^{\infty} e^{in\varphi}|n\rangle$, $|n\rangle$ being the Fock states. The exponential phase moments Ψ_k of this distribution are given by $\Psi_k = \int_{2\pi} e^{ik\varphi}\ p(\varphi)\ d\varphi$ and they can be expressed as

$$\Psi_k = \sum_{n=0}^{\infty} \varrho_{n+k,n} \tag{2}$$

for k positive and $\Psi_k = \Psi^*_{-k}$ for k negative. Our aim is to express Ψ_k by means of the measured quadrature distribution $p(x,\vartheta)$, i.e., in the form of Eq. (1) with $\mathcal{A} \equiv \Psi_k$. For this purpose we must find the corresponding kernel $K_k(x,\vartheta)$.

To do so, let us express the distribution $p(x,\vartheta)$ by means of the density matrix elements as

$$p(x,\vartheta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \psi_n(x)\psi_m(x)\varrho_{m,n}e^{i(n-m)\vartheta}. \tag{3}$$

Here $\psi_n(x)$ are the eigenfunctions of the harmonic oscillator Hamiltonian, $\psi_n(x) = (2^n n!\sqrt{\pi})^{-1/2}$ $\exp(-x^2/2)\ \mathrm{H}_n(x)$, $\mathrm{H}_n(x)$ being the Hermite polynomials. Substituting Eq. (3) into (1) and comparing with Eq. (2) we find that the kernels must be of the form $K_k(x,\vartheta) = e^{ik\vartheta}\ K_k(x)$, where the x-dependent part must satisfy the integral equation

$$2\pi \int_{-\infty}^{\infty} dx\, K_k(x)\psi_{n+k}(x)\psi_n(x) = 1 \tag{4}$$

$(n=0,1,2,\ldots)$. In [8] the solution of this equation is discussed in detail. Let us write here the result in the form

$$K_{2m}(x) = \frac{m!}{(2\pi)^{m+1}} \int_{-\infty}^{+\infty} dt_1\, e^{-t_1^2} \cdots$$
$$\times \cdots \int_{-\infty}^{+\infty} dt_{2m}\, e^{-2mt_{2m}^2} \left\{ \frac{\Phi[m+1, \frac{1}{2}, z_{2m}(1+z_{2m})^{-1}x^2]}{z_{2m}^m (1+z_{2m})^{m+1}} - \frac{1}{z_{2m}^m} \right\}, \tag{5}$$

$$K_{2m+1}(x) = \frac{2x(m+1)!}{(2\pi)^{m+3/2}} \int_{-\infty}^{+\infty} dt_1\, e^{-t_1^2} \cdots$$
$$\times \cdots \int_{-\infty}^{+\infty} dt_{2m+1}\, e^{-(2m+1)t_{2m+1}^2} \frac{\Phi[m+2, \frac{3}{2}, z_{2m+1}(1+z_{2m+1})^{-1}x^2]}{z_{2m+1}^m (1+z_{2m+1})^{m+2}}, \tag{6}$$

with $z_k = [\exp(-\sum_{j=1}^{k} t_j^2) - 1]/2$, $\Phi(a,b,y)$ being the confluent hypergeometric function.

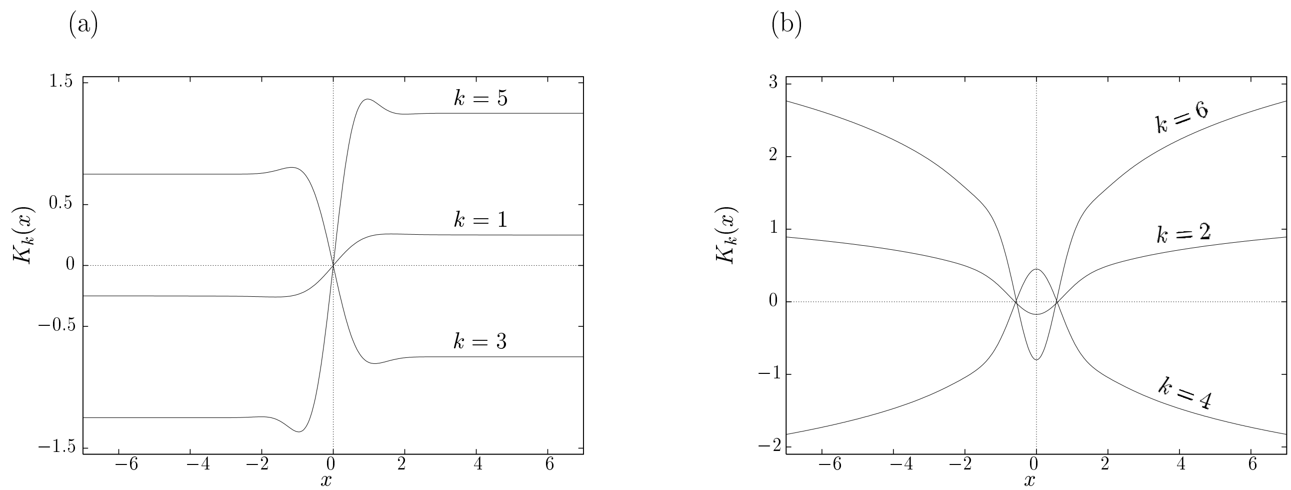

Figure 1: The functions $K_k(x)$.

In Fig. 1 we plot the functions $K_k(x)$ for several values of k. As can be seen, the functions $K_k(x)$ are well behaved and with increasing $|x|$ they quickly approach their (classical) asymptotics $K_k^c(x)$, where

$$K_{2m+1}^c(x) = \tfrac{1}{4}(-1)^m(2m+1)\,\mathrm{sign}\,(x) \tag{7}$$

and

$$K_{2m}^c(x) = \pi^{-1}(-1)^{m+1} m \ln|x| + C_{2m}, \tag{8}$$

C_{2m} being a (unimportant) constant. An essential difference between $K_k(x)$ and $K_k^c(x)$ appears only for x near zero, within the area of vacuum fluctuations. It is shown in [8] that the functions $K_k^c(x)$ can be obtained as kernels for sampling exponential phase moments in classical physics. Therefore, we have found kernels for direct sampling of the canonical phase moments which can be used for any state, regardless if it has typically quantum or classical properties.

3 Simulated measurements

To illustrate the applicability of the method we have performed computer simulations of homodyne measurements and used the kernels $K_k(x,\vartheta)$ to determine the moments Ψ_k. For comparison, we have also simulated double homodyne measurements to get exponential phase moments that correspond to the radially integrated Q function. In the computer simulations the state to be detected is the phase squeezed state $|\alpha, s\rangle$ with the coherent amplitude $\alpha = 5\times e^{i\varphi_0}$, $\varphi_0 = 0.6$, and the squeeze parameter $s = 6$ (the mean photon number of this state is $\langle n\rangle = 26.04$). For both the homodyne and the double homodyne measurements the total number of measurement events is $N_e = 6020$. For the homodyne measurements the LO phase ϑ takes 41 values equidistantly distributed over the 2π interval. As shown in [8], the statistical error of the sampled phase moments depends on the numbers of measurement events for different ϑ. By a proper distribution of the total number N_e for individual phases ϑ we can decrease the statistical error of various moments Ψ_k. In order to minimize the statistical error of the first moment Ψ_1, we increased the number of measurement events for such ϑ for which the peak of $p(x,\vartheta)$ is near $x = 0$ (maximum 800 events), whereas for ϑ yielding a peak far from zero the number of events was small (minimum 10 events).

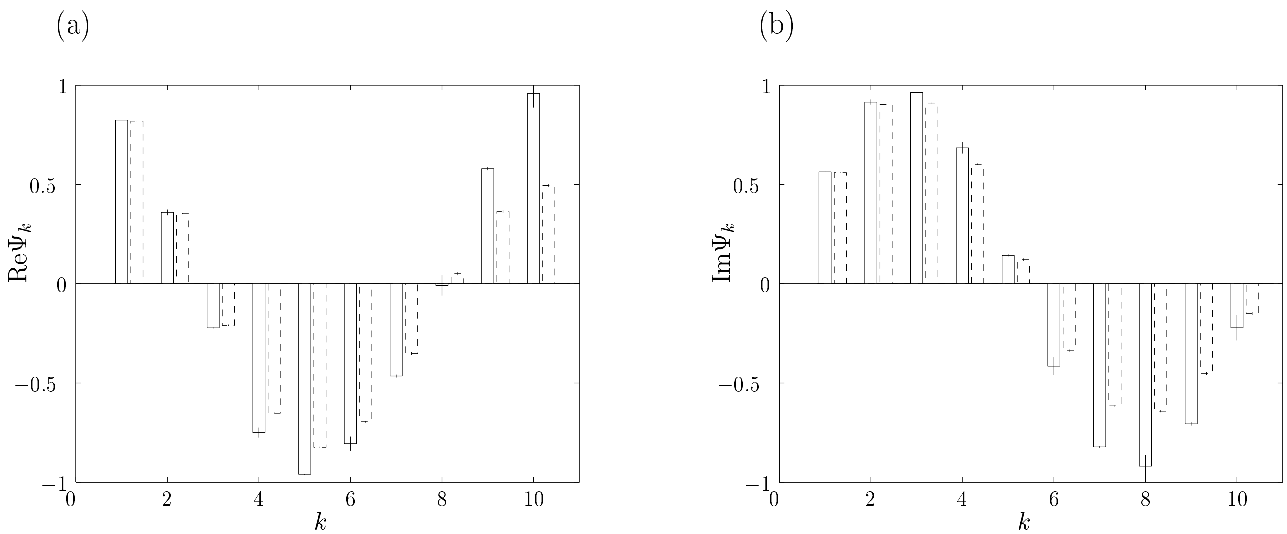

Figure 2: Real (a) and imaginary (b) parts of the experimentally determined exponential phase moments Ψ_k. Bars with full lines: direct sampling from balanced homodyning, bars with dashed lines: double homodyning. The vertical lines represent the estimated statistical error.

The experimentally determined phase moments together with the estimated statistical errors are shown in Fig. 2. We can see that the absolute values of moments of the integrated Q function are smaller than the corresponding values of the canonical distribution; the difference becomes larger with increasing k. This corresponds to the fact that the integrated Q function smears the structure of the canonical distribution.

Let us compare the statistical errors. For the odd moments the errors are approximately of the same magnitude in the two methods, whereas for the even moments the sampling method yields larger errors. This reflects the qualitatively different behavior of the kernels for k odd and k even. It is also related to the chosen numbers of measurement events for different ϑ: in our

example we have distributed the event numbers so as to minimize Ψ_1 which on the other hand increases the errors of even moments. Even though one could expect that the double homodyning - as a direct phase measurement - would yield generally smaller statistical errors, we find that the statistical error of the first moment Ψ_1 is smaller for the sampled canonical distribution. (The direct measurement means that a single measurement event yields a single value of phase.)

Let us mention that the first moment is connected to very important characteristics of the phase distribution. The mean value of phase $\bar{\varphi}$ can be calculated as $\bar{\varphi} = \arg\Psi_1$; this quantity can correspond, e.g., to a phase shift in an interferometer. Since Ψ_1 is determined more precisely in the balanced homodyning than in the double homodyning, the sampling method enables us to determine $\bar{\varphi}$ with smaller statistical error. From the experimental data we obtain $\bar{\varphi} = 0.5994 \pm 0.0011$ for the sampling method, whereas from the integrated Q function we obtain $\bar{\varphi} = 0.5990 \pm 0.0016$. (Note that for both distributions the correct value is $\bar{\varphi} = \varphi_0 = 0.6$.) As can be seen, the error of determination of the mean phase by means of homodyne sampling is about 70% of the error in the double homodyning. The moment Ψ_1 is also related to various phase uncertainties, which describe the "width" of the phase probability distribution. A phase uncertainty $\Delta\varphi$ can be defined as $\Delta\varphi = \arccos|\Psi_1|$, which is related to the Bandilla-Paul phase dispersion σ_{BP}^2 as $\sigma_{BP} = \sin\Delta\varphi$ and to the Holevo phase dispersion σ_H^2 as $\sigma_H = \tan\Delta\varphi$ [9]. (An advantage of the uncertainty $\Delta\varphi$ is that it enables us to measure the phase width in the same units as the phase itself - in radians, degrees, etc.) In this way we obtain $\Delta\varphi = 0.065$ for the canonical phase distribution and $\Delta\varphi = 0.125$ for the integrated Q function.

4 Discussion and conclusion

The presented method shows a very straightforward way for obtaining the exponential moments of the canonical phase from the data of homodyne detection. Direct sampling enables us to reconstruct the moments Ψ_k in real time as the experiment runs, together with the estimation of the statistical error [8]. In this way the theoretically profound concept of canonical phase can be connected with data obtained from present experiments.

The moments Ψ_k contain the same information as the probability distribution $p(\varphi)$. Therefore they can be used for reconstruction of the original function $p(\varphi)$. However, even the lowest moments give us an interesting information about the phase properties, e.g., the first moment Ψ_1 is directly related to the mean value of phase and to the phase uncertainty.

The integration kernels are well-behaved functions which rapidly approach their asymptotics given either as step-functions (for odd moments) or logarithmic functions (for even moments). As shown in [8], these functions can serve as kernels for sampling of the phase moments in classical physics. Therefore, we have found a unified approach which connects the measurement of the canonical phase with its classical counterpart.

It has been shown that the accuracy of the sampled moments of the canonical phase is comparable with that of the directly measured radially integrated Q function. Moreover, when the total number of measurement events is the same, the sampling method can yield the mean value of phase more precisely than the measurement of the Q function. Also this aspect can make the presented method very attractive for experimental applications.

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